AP-C Objectives (from College Board Learning Objectives for AP Physics)

▼ 1. Rotational Kinematics
- a. Understand and apply relationships between translational and rotational kinematics.
- b. Use the right hand rule to determine the direction of the angular velocity vector.

▼ 2. Moment of Inertia
- a. Determine by inspection which set of symmetrical objects of equal mass has the greatest moment of inertia.
- b. Determine by what factor an object's moment of inertia changes if its dimensions are increased by a consistent factor.
- c. Calculate the moment of inertia for a collection of point masses, a thin rod of uniform density, and a set of coaxial cylindrical shells.
- d. State and apply the parallel-axis theorem (PAT).

▼ 3. Torque
- a. Calculate the torque on a rigid object.
- b. Apply conditions of translational and rotational equilibrium to analyze a rigid object under the influence of coplanar forces applied at different locations.

▼ 4. Rotational Dynamics
- a. Determine the angular acceleration of an object when an external torque or force is applied.
- b. Determine the radial and tangential acceleration of a point on a rigid object.
- c. Analyze problems involving strings and massive pulleys.
- d. Analyze problems involving objects that roll with and without slipping.

▼ 5. Conservation of Energy with Rotation
- a. Apply conservation of energy to problems of fixed-axis rotation.
- b. Apply conservation of energy to objects undergoing translational and rotational motion.

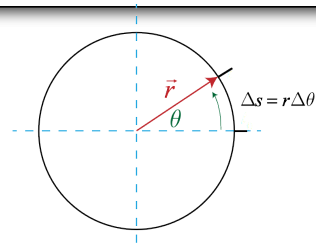
Rotational Kinematics

AP-C Objectives (from College Board Learning Objectives for AP Physics)

1. Rotational Kinematics
   - a. Understand and apply relationships between translational and rotational kinematics.
   - b. Use the right hand rule to determine the direction of the angular velocity vector.
   - c. Calculate the area under a force versus time graph and relate it to the change in momentum of an object.

Radians

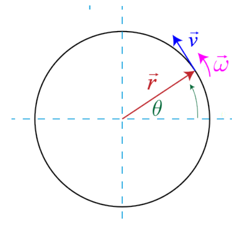
Angular displacements (Δθ) can be measured in degrees (°) or in radians. One revolution is 360°, or 2π radians. The distance around a circle, known as the circumference, is equal to 2π multiplied by the length of the radius. This length is typically written as Δs. You can convert from linear displacement to angular displacement using Δs=rΔθ.



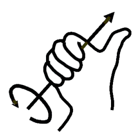
Velocity

Linear speed / velocity is given by **v**. Angular speed / velocity is given by **ω**.

$$\vec{v} = \frac{d\vec{s}}{dt} \qquad \vec{\omega} = \frac{d\vec{\theta}}{dt}$$



Direction of Angular Velocity Vector given by Right Hand Rule



Conversion Between Translational and Rotational

Variable	Translational	Angular
Displacement	$s = r\theta$	$\theta = \frac{s}{r}$
Velocity	$v = r\omega$	$\omega = \frac{v}{r}$
Acceleration	$a = r\alpha$	$\alpha = \frac{a}{r}$
Time	t	t

Translational vs. Rotational Variables

Variable	Translational	Angular
Displacement	Δs	Δθ
Velocity	v	ω
Acceleration	a	α
Time	t	t

Translational	Rotational
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$\Delta x = v_0 t + \frac{1}{2}at^2$	$\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a\Delta x$	$\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$

A frog rides a unicycle. If the unicycle wheel begins at rest, and accelerates uniformly in a clockwise direction to an angular velocity of 15 rpms in a time of 6 seconds, find the angular acceleration of the unicycle wheel.

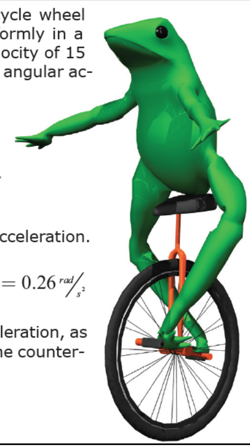
First, convert 15 rpms to rads/s.

$$\frac{15 \text{ revs}}{\text{min}} \times \frac{1 \text{ min}}{60s} \times \frac{2\pi \text{ rad}}{1 \text{ rev}} = 1.57 \,{}^{rad}\!/_{s}$$

Next, use the definition of angular acceleration.

$$\alpha = \frac{\Delta\omega}{t} = \frac{\omega - \omega_0}{t} = \frac{(1.57 \,{}^{rad}\!/_{s})\text{-}0}{6s} = 0.26 \,{}^{rad}\!/_{s^2}$$

Again, note the positive angular acceleration, as the bicycle wheel is accelerating in the counter-clockwise direction.



A compact disc player is designed to vary the disc's rotational velocity so that the point being read by the laser moves at a linear velocity of 1.25 m/s. What is the CD's rotational velocity in revs/s when the laser is reading information on an inner portion of the disc at a radius of 0.03m?

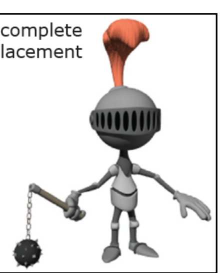
$$\omega = \frac{v}{r} = \frac{1.25 \,{}^{m}\!/_{s}}{0.03m} = 41.7 \,{}^{rad}\!/_{s}$$

$$\frac{41.7 \text{ rad}}{s} \times \frac{1 \text{ rev}}{2\pi \text{ rad}} = 6.63 \,{}^{rev}\!/_{s}$$

A knight swings a mace of radius 1m in two complete revolutions. What is the translational displacement of the mace?

(1) 3.1 m
(2) 6.3 m
(3) 12.6 m
(4) 720 m

(3) $s = r\theta = (1m)(4\pi \text{ rads}) = 12.6m$



Moment of Inertia

AP-C Objectives (from College Board Learning Objectives for AP Physics)

▼ 1. Moment of Inertia
- a. Determine by inspection which set of symmetrical objects of equal mass has the greatest moment of inertia.
- b. Determine by what factor an object's moment of inertia changes if its dimensions are increased by a consistent factor.
- c. Calculate the moment of inertia for a collection of point masses, a thin rod of uniform density, and a set of coaxial cylindrical shells.
- d. State and apply the parallel-axis theorem (PAT).

▼ Types of Inertia

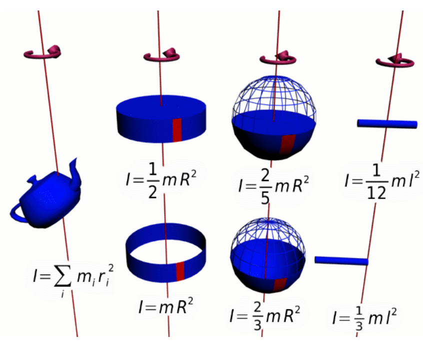
Inertial mass, also known as **translational inertia**, (m), is an object's ability to resist a linear acceleration, which correlates to how much "stuff" makes up an object.

Moment of Inertia, also known as **rotational inertia** (I), describes an object's resistance to a rotational acceleration.

Objects that have most of their mass near their axis of rotation have a small rotational inertia, while objects that have more mass farther from the axis of rotation have larger rotational inertias.

Calculating Moment of Inertia (I)

$$I=\sum mr^2=\int r^2\,dm$$



▼ Moment of Inertia of Point Masses

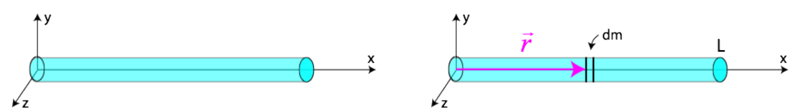
Find the moment of inertia of two 5 kg bowling balls joined by a meter-long rod of negligible mass when rotated about the center of the rod. Compare this to the moment of inertia of the object when rotated about one of the masses.

$$I=\sum mr^2=m_1r_1^2+m_2r_2^2$$
$$I=5\times0.5^2+5\times0.5^2$$
$$I=2.5\ kg\bullet m^2$$

$$I=\sum mr^2=m_1r_1^2+m_2r_2^2$$
$$I=5\times1^2+5\times0^2$$
$$I=5\ kg\bullet m^2$$

▼ Moment of Inertia of a Uniform Rod About Its End and About Its Center

Find the moment of inertia of a uniform rod about its end, compare to the moment of inertia of the rod about its center.



Define a linear mass density, λ, equal to the total mass M divided by the length L. Then, dm=λdx.

Rotating About the End

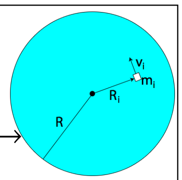
$$I=\int r^2\,dm\xrightarrow{dm=\lambda dx}I=\int_{x=0}^{x=L}x^2\lambda\,dx\rightarrow$$
$$I=\lambda\int_0^L x^2\,dx=\lambda\frac{L^3}{3}\xrightarrow{\lambda=\frac{M}{L}}I=\frac{M}{L}\frac{L^3}{3}\rightarrow$$
$$I=\tfrac{1}{3}ML^2$$

Rotating About the Center

$$I=\int r^2\,dm\xrightarrow{dm=\lambda dx}I=\int_{x=-\frac{L}{2}}^{x=\frac{L}{2}}x^2\lambda\,dx\rightarrow$$
$$I=\lambda\int_{-\frac{L}{2}}^{\frac{L}{2}}x^2\,dx=\frac{\lambda}{3}\left(\frac{L^3}{8}-\frac{-L^3}{8}\right)=\frac{\lambda}{3}\frac{L^3}{4}\xrightarrow{\lambda=\frac{M}{L}}$$
$$I=\left(\frac{M}{L}\right)\left(\frac{L^3}{12}\right)\rightarrow I=\frac{ML^2}{12}$$

KE of a Rotating Disc

$$K_i=\tfrac{1}{2}m_iv_i^2\xrightarrow{v=\omega R}K_i=\tfrac{1}{2}m_i\omega^2r_i^2$$
$$K_{total}=\sum_i K_i=\frac{\omega^2}{2}\sum_i m_ir_i^2\xrightarrow{I=\sum mr^2}$$
$$K_{total}=\tfrac{1}{2}I\omega^2$$

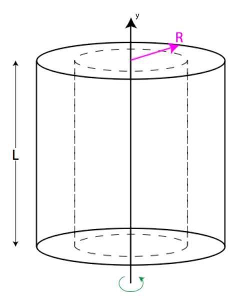


▼ Moment of Inertia of a Solid Cylinder

Find the moment of inertia of a uniform solid cylinder about its axis.

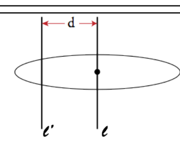
Define a volume mass density, ρ, as the total mass divided by the volume of the cylinder: ρ=M/(πR²L). A differential of mass, then, is the mass of a thin, hollow cylindrical shell of width dr and area 2πrL. The total differential of mass, then, is the volume of the hollow cylindrical shell multiplied by the volume mass density, so dm=2πrL(dr)(ρ).

$$I=\int r^2\,dm\xrightarrow{dm=2\pi rL(dr)(\rho)}I=\int_{r=0}^{R}r^2(2\pi\rho Lr\,dr)=$$
$$2\pi\rho L\int_0^R r^3\,dr=2\pi\rho L\frac{R^4}{4}\xrightarrow{\rho=\frac{M}{\pi R^2L}}I=\frac{2\pi LMR^4}{4\pi R^2L}\rightarrow$$
$$I=\tfrac{1}{2}MR^2$$



▼ Parallel Axis Theorem (PAT)

If you know the moment of inertia of any object through an axis intersecting the center of mass of the object (I), then you can find the moment of inertia around any axis parallel to I.



$$I_{l'}=I_l+md^2$$

▼ Example: Find Moment of Inertia of a Rod About One End using PAT

Given the moment of inertia of a rod around its center (I=ML²/12), use the PAT to find the moment of inertia of a rod about its end.

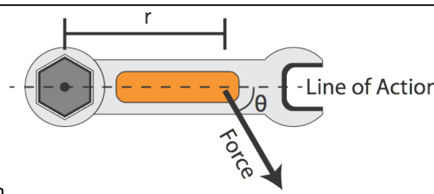
$$I_{end}=I_{center}+md^2=\frac{ML^2}{12}+M\left(\frac{L}{2}\right)^2\rightarrow$$
$$I_{end}=\frac{ML^2}{12}+\frac{ML^2}{4}=\frac{ML^2}{3}$$

AP-C Objectives (from College Board Learning Objectives for AP Physics)

1. Torque
   - a. Calculate the torque on a rigid object.
   - b. Apply conditions of translational and rotational equilibrium to analyze a rigid object under the influence of coplanar forces applied at different locations.

Torque

Torque (τ) is a force that causes an object to turn. It must be perpendicular to the displacement to cause a rotation, and the further it is away from the point of rotation, the more leverage you obtain, so this distance is known as the lever arm (r).



The direction of the torque vector is perpendicular to both the position vector and the force vector. You can find the direction using the right-hand rule. Point the fingers of your right hand in the direction of the line of action, and bend your fingers in the direction of the force. Your thumb then points in the direction of your thumb. Note that positive torques are cause counter-clockwise rotations, and negative torques cause clockwise.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\left|\vec{\tau}\right| = rF\sin\theta$$

Translational vs. Rotational N2Law

$$\vec{F}_{net} = m\vec{a}$$

$$\vec{\tau}_{net} = I\vec{\alpha}$$

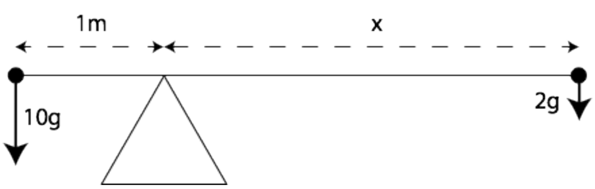
Equilibrium

Static equilibrium implies that the net force and the net torque are zero, and the system is at rest.

Dynamic equilibrium implies that the net force and the net torque are zero, and the system is moving at constant translational and rotational velocity.

See Saw Sample Problem

A 10-kg tortoise sits on a see-saw 1 meter from the fulcrum. Where must a 2-kg hare sit in order to maintain static equilibrium? What is the force on the fulcrum?

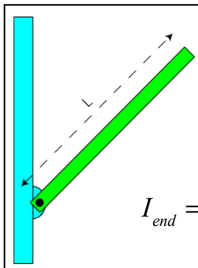


$$\vec{\tau}_{net} = I\vec{\alpha} = 0 \rightarrow$$
$$(10g)(1m) - (2g)x = 0 \rightarrow$$
$$x = 5m$$

$$\vec{F}_{net} = m\vec{a} = 0 \rightarrow$$
$$-10g + F_{fulcrum} - 2g = 0 \rightarrow$$
$$F_{fulcrum} = 12g = 120N$$

Beam Sample Problem

A beam of mass M and length L has a moment of inertia about its center of $ML^2/12$. The beam is attached to a frictionless hinge at an angle of 45° and allowed to swing freely. Find the beam's angular acceleration.

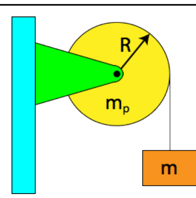


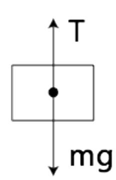
$$I_{end} = I_{cm} + md^2 \xrightarrow{PAT} I_{end} = \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2 = \frac{ML^2}{3}$$

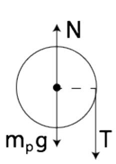
$$\tau_{net} = I\alpha \rightarrow -Mg\cos\theta\left(\frac{L}{2}\right) = I\alpha \rightarrow \alpha = \frac{-Mg\cos\theta L}{2I} \xrightarrow{I=\frac{ML^2}{3}} \alpha = \frac{-3g\cos\theta}{2L}$$

Non-Ideal Pulley with Mass Problem

A light string attached to a mass m is wrapped around a pulley of mass m_p and radius R. Find the acceleration of the mass.







$$mg - T = ma \xrightarrow{T=\frac{1}{2}m_p a} mg - \tfrac{1}{2}m_p a = ma \rightarrow$$

$$mg = a\left(m + \frac{m_p}{2}\right) \rightarrow a = \frac{mg}{m + \frac{m_p}{2}}$$

$$\tau_{net} = I\alpha \xrightarrow[I=\frac{1}{2}m_p R^2]{\tau=RT} RT = \left(\tfrac{1}{2}m_p R^2\right)\alpha \rightarrow$$

$$T = \tfrac{1}{2}m_p R\alpha \xrightarrow{a=R\alpha} T = \tfrac{1}{2}m_p a$$

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1. Rotational Dynamics
   - a. Determine the angular acceleration of an object when an external torque or force is applied.
   - b. Determine the radial and tangential acceleration of a point on a rigid object.
   - c. Analyze problems involving strings and massive pulleys.
2. Conservation of Energy with Rotation
   - a. Apply conservation of energy to problems of fixed-axis rotation.
   - b. Apply conservation of energy to objects undergoing translational and rotational motion.

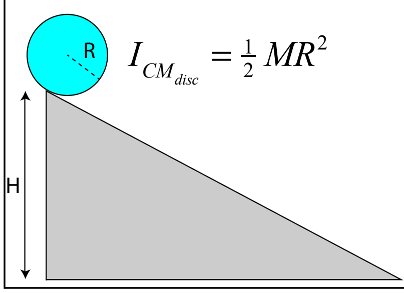
Conservation of Energy

Because objects may have both translational and rotational kinetic energy, we must now take into account that the total kinetic energy of an object is the sum of its translational kinetic energy (use the velocity of the center of mass) and its rotational kinetic energy (using the moment of inertia about its point of rotation).

$$K_{translational} = \tfrac{1}{2}mv^2$$
$$K_{rotational} = \tfrac{1}{2}I\omega^2$$
$$K_{total} = K_{translational} + K_{rotational}$$

Energy Conservation with Rolling

Find the speed of a disc of radius R which starts at rest and rolls down an incline of height H.



$$I_{CM_{disc}} = \tfrac{1}{2}MR^2$$

$$K_i + U_i = K_f + U_f \xrightarrow[U_f=0]{K_i=0} U_i = K_f \rightarrow MgH = \tfrac{1}{2}Mv_{CM}^2 + \tfrac{1}{2}I\omega^2 \rightarrow$$

$$MgH = \tfrac{1}{2}Mv_{CM}^2 + \tfrac{1}{2}(\tfrac{1}{2}MR^2)\omega^2 \rightarrow gH = \frac{v_{CM}^2}{2} + \frac{1}{4}\omega^2 R^2 \xrightarrow{v=\omega R}$$

$$gH = \frac{v_{CM}^2}{2} + \frac{v_{CM}^2}{4} \rightarrow gH = \frac{3}{4}v_{CM}^2 \rightarrow v_{CM}^2 = \frac{4}{3}gH \rightarrow v_{CM} = \sqrt{\tfrac{4}{3}gH}$$

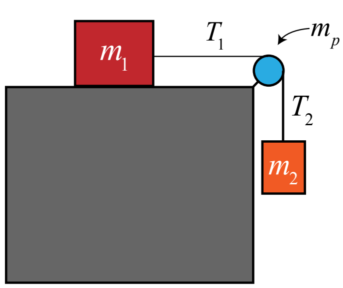
Rotational Dynamics

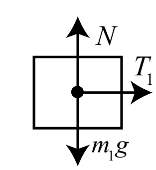
Much like translational dynamics, rotational dynamics is governed by Newton's 2nd Law of Motion. However, instead of net force equal to the product of the object's mass and acceleration, the net torque is equal to the product of the object's moment of inertia and angular acceleration.

$$\vec{F}_{net} = m\vec{a}$$
$$\vec{\tau}_{net} = I\vec{\alpha}$$

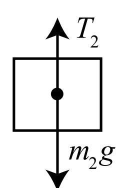
Strings with Massive Pulleys

Two blocks are connected by a light string over a pulley of mass m_p and radius R. Find the acceleration of mass m_2 if m_1 sits on a frictionless surface.

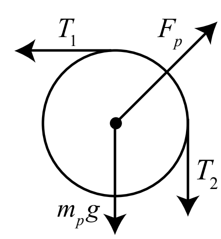




$$F_{net_X} = T_1 = m_1a$$



$$F_{net_Y} = m_2g - T_2 = m_2a$$
$$T_2 = m_2g - m_2a$$



$$\tau_{net} = T_2R - T_1R = I\alpha$$

$$\tau_{net} = T_2R - T_1R = I\alpha \xrightarrow[T_2=m_2g-m_2a]{T_1=m_1a} R(m_2g - m_2a - m_1a) = I\alpha \xrightarrow[\alpha=\frac{a}{R}]{I_{disc}=\frac{1}{2}m_pR^2}$$

$$R(m_2g - m_2a - m_1a) = \tfrac{1}{2}m_pR^2\left(\frac{a}{R}\right) = \tfrac{1}{2}m_paR \rightarrow m_2g - m_2a - m_1a = \tfrac{1}{2}m_pa \rightarrow$$

$$m_2g = m_1a + m_2a + \tfrac{1}{2}m_pa \rightarrow m_2g = a(m_1 + m_2 + \tfrac{1}{2}m_p) \rightarrow a = \frac{m_2g}{m_1 + m_2 + \tfrac{1}{2}m_p}$$

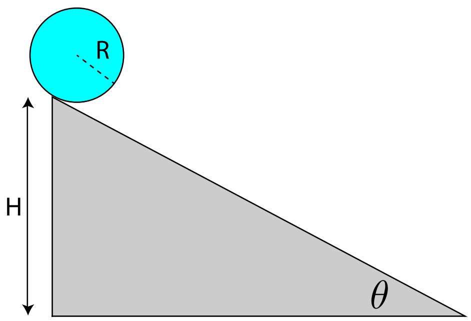
AP-C Objectives (from College Board Learning Objectives for AP Physics)

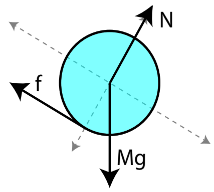
▼ 1. Rotational Dynamics

- a. Analyze problems involving objects that roll with and without slipping.

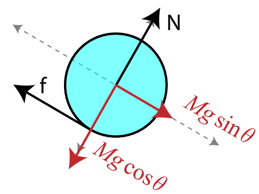
▼ Rolling Without Slipping

A disc of radius R rolls down an incline of angle θ without slipping. Find the force of friction on the disc.





$$F_{net_Y} = N - Mg\cos\theta = 0$$
$$\rightarrow N = Mg\cos\theta$$



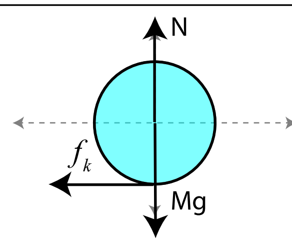
$$F_{net_X} = Mg\sin\theta - f = Ma$$

$$\tau_{net} = fR = I\alpha \xrightarrow{I=\frac{1}{2}MR^2} fR = \frac{MR^2\alpha}{2} \xrightarrow{a=R\alpha} fR = \frac{MRa}{2} \rightarrow f = \frac{Ma}{2} \xrightarrow{Ma=Mg\sin\theta - f}$$

$$f = \frac{Mg\sin\theta - f}{2} \rightarrow 2f = Mg\sin\theta - f \rightarrow 3f = Mg\sin\theta \rightarrow f = \frac{Mg\sin\theta}{3}$$

▼ Rolling With Slipping

A bowling ball of mass M and radius R skids horizontally down the alley with an initial velocity of v_0. Find the distance the ball skids before rolling given a coefficient of kinetic friction of μ_k.



$$\boxed{\begin{aligned} &F_{net_X} = -f_k = Ma \\ &F_{net_Y} = N - Mg = 0 \rightarrow N = Mg \\ &f_k = \mu_k N \xrightarrow{N=Mg} f_k = \mu_k Mg \end{aligned}}$$

$$\boxed{-f_k = Ma \xrightarrow{f_k=\mu_k g} -\mu_k Mg = Ma \rightarrow a = -\mu_k g}$$

$$\tau_{net} = f_k R = I\alpha \xrightarrow[I=\frac{2}{5}MR^2]{f_k=\mu_k Mg} \mu_k MgR = \tfrac{2}{5}MR^2\alpha \rightarrow \mu_k g = \tfrac{2}{5}R\alpha \rightarrow \alpha = \frac{5\mu_k g}{2R}$$

$$v = v_0 + at \xrightarrow{a=-\mu_k g} v = v_0 - \mu_k gt$$

Stops slipping at this condition

$$\omega = \omega_0 + \alpha t \xrightarrow[\omega_0=0]{\alpha=\frac{5\mu_k g}{2R}} \omega = \frac{5\mu_k g}{2R}t$$

$$v = R\omega$$

$$v_0 - \mu_k gt = R\left(\frac{5\mu_k g}{2R}t\right)$$

$$v_0 - \mu_k gt = \frac{5\mu_k g}{2}t$$

$$v_0 - \mu_k gt = \frac{5\mu_k g}{2}t \rightarrow v_0 = \mu_k gt + \frac{5\mu_k g}{2}t = \tfrac{7}{2}\mu_k gt \rightarrow t = \frac{2v_0}{7\mu_k g}$$

$$\boxed{\Delta x = x_0 + v_0 t + \tfrac{1}{2}at^2 = v_0\left(\tfrac{2v_0}{7\mu_k g}\right) + \tfrac{1}{2}(-\mu_k g)\left(\tfrac{2v_0}{7\mu_k g}\right)^2 \rightarrow \Delta x = \frac{12v_0^2}{49\mu_k g}}$$